Genus 3 covers of elliptic curves

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2. $\mathrm{Jac}(C) \sim E \times \mathrm{Jac}(X)$ with X of genus 2

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Remark

Finding E_2, E_3 is as hard as finding E, and we know how to do that.

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3. Prym-like varieties

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If $\mathrm{Aut}(C) = \mathbb{Z}/2\mathbb{Z}$ *one can explicitly write down a genus 2 curve* X *such that* $\mathrm{Jac}(C) \sim E \times \mathrm{Jac}(X)$*.* X *is defined over the same field as* C*.*

Galois cases

- Suppose $C \to E$ is Galois with "large" automorphism group – i.e. D_4, Q_8, S_3. Then $\mathrm{Jac}(C)$ is the product of three elliptic curves.

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- When the group is $\mathbb{Z}/3\mathbb{Z}$, the abelian surface has QM

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Question

Let Θ be the theta divisor of Jac(C). What is the degree of the polarization $\iota_A^* \Theta$?

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Question

Is the isogeny defined over the same field as $C \to E$?

Reconstruction: analytic approach

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- Determine an isogeny $A \to A'$ with A' principally polarized
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- Reconstruct X from A' (Guàrdia)